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The sensitivity analysis of population projections

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1 Introduction

Nathan Keyfitz (1964), in an important early contribution, noted that population projections based on the cohort component method can be written as matrix population models. This paper presents the first steps towards a perturbation analysis of population projections, using matrix calculus methods.

We use the term *projection* to describe conditional predictions of population size and structure, such as are regularly developed by national governments, international consortia (e.g., Eurostat), and non-governmental organizations (U.N.). All projections are conditional in the sense that they are based on a hypothetical scenario defining future rates of mortality, fertility, and migration (collectively, the "vital rates"), and also conditional on an initial population. The vital rate scenarios can be described in terms of a set of parameters, and the question arises how the results of the projection would change in response to changes in the parameters. Demographers have become increasingly concerned with estimating this uncertainty of projection results ex ante or ex post (Booth 2006, Ahlburg and Lutz 1998).

This is a question of sensitivity analysis and our approach is to calculate the derivatives of the projection results to the parameters and initial conditions. This gives the effects of small changes, gives approximate results for quite large changes, and identifies parameters with particularly large or small impacts on the results. Perturbation analysis is useful because:

1. It can project the consequences of future changes, including human actions, both intentional (e.g., policies to encourage reproduction, public health interventions, or conservation strategies applied to endangered species) or unintentional (e.g., consequences of pollution or environmental degredation).
2. It can be used to compare the results of policy interventions and identify interventions that will have particularly large effects.
3. It can be used retrospectively to explain the causes of observed changes.
4. It can be used to identify parameters that should be estimated with high accuracy, because they have large effects on the results.
5. It increases the understanding of the processes determining the outcome.

Our approach here is to write the projection as a matrix operator, and then to use matrix calculus techniques (e.g., Caswell 2007, 2008, 2012) to derive the needed derivatives of the results to underlying parameters. These methods are easily implemented in any matrix-oriented computer language, especially MATLAB, but also R.

1.1 Projection

Any cohort-component population projection can be written as a matrix operator:

$$\mathbf{n}(t+1) = \mathbf{A}[t]\mathbf{n}(t) + \mathbf{b}(t) \qquad \mathbf{n}(0) = \mathbf{n}_0 \tag{1}$$

where $\mathbf{n}(t)$ is a vector giving the numbers of individuals by age class or stage at time t, $\mathbf{A}(t)$ is a projection matrix incorporating the vital rates at time t, and $\mathbf{b}(t)$ is a vector giving immigration at time t. The projection begins with a specified initial condition, denoted $\mathbf{n}_0$, and is carried out until some target time T.

The formulation (1) is general enough to encompass all the projections typically used. The vector $\mathbf{n}$ can incorporate any type of population structure considered relevant. If individuals are grouped into age classes, then $\mathbf{A}$ is the familiar Leslie matrix, with survival probabilities on the subdiagonal, fertilities in the first row, and zeros elsewhere. If individuals are classified by other criteria ("stages" in common usage), $\mathbf{A}$ will have the structure needed to capture transitions among stages based on physiological condition, developmental stage, socio-economic grouping, marital status, parity status, etc. When individuals are classified by multiple criteria (multi-state models), the matrix $\mathbf{A}$ can be constructed from the transitions involving each criterion using the vec-permutation matrix (Caswell 2009, 2012, Hunter and Caswell 2005).

Expressing the projection as a matrix operator has the advantage of focusing attention on its mathematical structure, and thus inviting further analyses. In practice, official projections are often implemented as computer algorithms, the details of which permit almost endless fine-tuning of relationships. In principle, however, any such program can be written in the form of (1).

The future being unknown, considerable ingenuity is exercised in constructing $\mathbf{A}[t]$ and $\mathbf{b}(t)$, in order to capture the dynamics of the vital rates in the future. Examples include assumed scenarios (e.g., the convergence scenarios used by Eurostat), extrapolation of trends (including sophisticated time-series models), and dependence on external factors that can in turn be forecast (e.g., effects of climate change on vital rates and hence on future population (e.g., Hunter et al. 2010, Jenouvrier et al. 2009, 2012).

1. Assumed scenarios. Possible future trends in vital rates have been defined by drawing on experts opinions. The experts may be directly involved in preparing the projections by choosing model, parameters and assumptions. The projections of Eurozone countries by Eurostat, for example, are based on an assumption of convergence of the mortality and fertility of all European countries to a common value in the long run, i.e. until 2150(Lanzieri: Europop2008). The rates for a given country in each year are determined by interpolating between the rates at the start of the projection and the final target rates. Other studies have been based on the opinion of experts who are not directly involved in projection process. Lutz and colleagues for instance have proposed a Delphi-method based approach to collect and aggregate external experts opinions on demographic trends in a systematic manner (Ahlburg and Lutz 1998). Expectations of population members about their own lives (e.g. survey data on the expected number of children or expected remaining life expectancy) have also been used to define scenarios. Regardless of the methods used, these approaches all produce trajectories of mortality, fertility, and migration, over the projection horizon, reflecting some idea of how the future might unfold.

2. Extrapolation of trends. This approach relies on the observation that some vital rates (particularly mortality and fertility rates) develop gradually over time. Methods based on extrapolation, such as the Lee-Carter model or ARIMA modeling, assume that trends observed in the past will be similar in the future.

3. Dependence on other factors, which can themselves be projected. For example, projections of populations of polar bears and emperor penguins under the impact of climate change have been based on projections of sea ice conditions (a critical environmental variable for these species) generated by models of global climate conditions produced by the IPCC (Hunter et al. 2010, Jenouvrier et al. 2009, 2012). Similarly, projections of human populations have been based on expectations about future economic, social or environmental developments (Booth 2006).

4. Mixture of approaches. Finally, the approaches have not only been used individually. Many studies combine different approaches, for instance by extrapolating past trends and using experts opinions to take possible future structural changes into account.

Any projection depends on a set of parameters which jointly determine the matrices $\mathbf{A}[t]$ and the immigration vectors $\mathbf{b}[t]$. We will write this set of parameters as a vector $\boldsymbol{\theta}$, assumed to be of dimension p. Our approach is flexible enough to address almost any kind of parametrization. Parameters may be expressed as events, probabilities or rates, and they may refer to cohorts or periods. They may be aggregate measures or they may be disaggregated by age, sex or other characteristics. Parameters may also be further decomposed, for instance, assumptions on fertility may be defined by parity (Booth 2006).

Perhaps the simplest case is concerned with the parameters

$$\boldsymbol{\mu}(t) \quad = \quad \text{vector of mortality rates} \tag{2}$$

$$\begin{aligned} \mathbf{f}(t) &= \text{vector of age-specific fertility} \qquad (3)\\ \mathbf{b}(t) &= \text{immigration vector} \qquad (4) \end{aligned}$$

However, these vectors might be expressed as functions of a scalar quantity such as life expectancy, or a parametric model such as the Gompertz, gamma-Gompertz, or Siler models for mortality, or the Coale-Trussel function for fertility.

To explicitly represent the parameterization, we will write the projection (1) as

$$\mathbf{n}(t+1) = \mathbf{A}[t, \boldsymbol{\theta}]\mathbf{n}(t) + \mathbf{b}[t, \boldsymbol{\theta}] \qquad (5)$$

2 Perturbation analysis of projections

Our goal is to compute the effect of changes in these parameters on the results of the projection. We do so using the matrix calculus approach introduced to demography in a series of papers (Caswell 2007, 2008, 2010, 2011, 2012a,b, Caswell and Shyu 2012). Matrix calculus methods are outlined in mathematical terms by Magnus and Neudecker (1987) and in demographic terms by Caswell (2007, 2010).

2.1 Differentiation of the projected population

Let $\boldsymbol{\theta}(t)$ be the parameter vector; we will consider later the specific cases where the parameters of interest are mortality, fertility, or immigration. Our goal is to find the effect of changes in $\boldsymbol{\theta}(x)$ on $\mathbf{n}(t)$, for $x = 0, \ldots, T$ and $t = 0, \ldots, T$. Some results are trivial; e.g., changes in $\boldsymbol{\theta}(x)$ can have no effect on $\mathbf{n}(t)$ for $t < x$ (we ignore the complications of time travel). However, a perturbation at some time x will ripple through $\mathbf{n}(t)$ for all $t > x$, and our goal is to find out how.

Differentiate (5), obtaining

$$d\mathbf{n}(t+1) = (d\mathbf{A}[t])\,\mathbf{n}(t) + \mathbf{A}[t]d\mathbf{n}(t) + d\mathbf{b}(t) \qquad (6)$$

Applying the vec operator yields

$$d\mathbf{n}(t+1) = \left(\mathbf{n}^{\mathsf{T}}(t) \otimes \mathbf{I}\right) d\mathrm{vec}\,\mathbf{A}[t] + \mathbf{A}[t]d\mathbf{n}(t) + d\mathbf{b}(t) \qquad (7)$$

By the chain rule for matrix calculus,

$$\frac{d\mathbf{n}(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \left(\mathbf{n}^{\mathsf{T}}(t) \otimes \mathbf{I}\right) \frac{d\mathrm{vec}\,\mathbf{A}[t]}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \mathbf{A}[t]\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \frac{d\mathbf{b}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \qquad (8)$$

This is a dynamic system in $d\mathbf{n}(t)/\boldsymbol{\theta}^{\mathsf{T}}(x)$, that can be iterated from the initial condition

$$\frac{d\mathbf{n}(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = 0 \qquad (9)$$

on the grounds that parameter changes do not affect the initial population vector.

The derivatives of $\mathbf{A}$ and $\mathbf{b}$ in (8) are as follows.

When $\boldsymbol{\theta} = \boldsymbol{\mu}$: Define the indicator matrix $\mathbf{Z}$ with ones on the subdiagonal and zeros elsewhere, and the survival vector $\mathbf{p} = \exp(-\boldsymbol{\mu})$. Let $\delta(m, n)$ be the Kronecker delta function

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad (10)$$

Then

$$\frac{d\text{vec}\,\mathbf{A}[t]}{d\boldsymbol{\mu}^{\mathsf{T}}(x)} = -\delta(x,t)\text{diag}\left(\text{vec}\,\mathbf{Z}\right)\left(\mathbf{1}\otimes\mathbf{I}\right)\text{diag}\left(\mathbf{p}\right) \tag{11}$$

$$\frac{d\mathbf{b}(t)}{d\boldsymbol{\mu}^{\mathsf{T}}(x)} = 0 \tag{12}$$

where $\mathbf{1}$ is a vector of ones.

When $\boldsymbol{\theta} = \mathbf{f}$: In this case,

$$\frac{d\text{vec}\,\mathbf{A}[t]}{d\mathbf{f}^{\mathsf{T}}(x)} = -\delta(x,t)\left(\mathbf{I}\otimes\mathbf{e}_1\right) \tag{13}$$

$$\frac{d\mathbf{b}(t)}{d\boldsymbol{\mu}^{\mathsf{T}}(x)} = 0 \tag{14}$$

where $\mathbf{e}_1$ is the first unit vector.

When $\boldsymbol{\theta} = \mathbf{b}$: When the parameter vector is the immigration vector, then

$$\frac{d\text{vec}\,\mathbf{A}[t]}{d\mathbf{f}^{\mathsf{T}}(x)} = 0 \tag{15}$$

$$\frac{d\mathbf{b}(t)}{d\boldsymbol{\mu}^{\mathsf{T}}(x)} = \delta(x,t)\mathbf{I} \tag{16}$$

These results are extremely general. They provide the sensitivity of

- every age class,
- at every time from 0 to T,
- with respect to changes in $\left\{\begin{array}{c}\text{mortality}\\ \text{fertility}\\ \text{immigration}\end{array}\right\}$ of every age class,
- at every time from 0 to T

2.2 Two-sex projections

Projections of populations by age and sex can be written in several ways as generalizations of (1). Here we will use the following approach. Define $\mathbf{n}_f$ and $\mathbf{n}_m$ as the female and male population vectors. Decompose the projection matrix for females into

$$\mathbf{A}_f = \mathbf{U}_f + \mathbf{F}_f \tag{17}$$

where $\mathbf{U}$ describes transitions and survival of extant individuals and $\mathbf{F}$ describes the production of new individuals by reproduction. There exists a matrix $\mathbf{U}_m$ for transitions and survival of males, but reproduction comes from females and hence there is no $\mathbf{F}_m$. The projection model becomes

$$\mathbf{n}_f(t+1) = \Big(\mathbf{U}_f + (1-r)\mathbf{F}_f\Big)\,\mathbf{n}_f(t) \tag{18}$$

$$\mathbf{n}_m(t+1) = \mathbf{U}_m\mathbf{n}_m(t) + r\mathbf{F}_f\mathbf{n}_f(t) \tag{19}$$

To obtain the sensitivity of the female projection, differentiate both sides of (18), to obtain

$$d\mathbf{n}_f(t+1) = (d\mathbf{U}_f)\,\mathbf{n}_f(t) + \mathbf{U}_f d\mathbf{n}_f(t) + (1-r)\,(d\mathbf{F})\,\mathbf{n}_f. \tag{20}$$

Apply the vec operator to both sides,

$$\begin{aligned} d\mathbf{n}_f(t+1) &= \left(\mathbf{n}_f^{\mathsf{T}}(t)\otimes\mathbf{I}\right) d\mathrm{vec}\,\mathbf{U}_f + \mathbf{U}_f d\mathbf{n}_f(t) + (1-r)\left(\mathbf{n}_f^{\mathsf{T}}\otimes\mathbf{I}\right) d\mathrm{vec}\,\mathbf{F} + (1-r)\mathbf{F}d\mathbf{n}_f(t) & (21)\\ &= \mathbf{A}_f(t)d\mathbf{n}_f(t) + \left(\mathbf{n}_f^{\mathsf{T}}(t)\otimes\mathbf{I}\right) d\mathrm{vec}\,\mathbf{A}_f(t) & (22)\end{aligned}$$

This is recognizable as the same as the sensitivity projection (8), but with fertility modified by the sex ratio.

Following the same procedure for males, we differentiate (19), and obtain

$$\begin{aligned} d\mathbf{n}_m(t) &= (d\mathbf{U}_m)\,\mathbf{n}_m(t) + \mathbf{U}_m\,(d\mathbf{n}_m(t)) + rd\,(d\mathbf{F})\,\mathbf{n}_f(t) + r\mathbf{F}d\mathbf{n}_f(t) & (23)\\ &= \mathbf{U}_m d\mathbf{n}_m(t) + \left(\mathbf{n}_m^{\mathsf{T}}(t)\otimes\mathbf{I}\right) d\mathrm{vec}\,\mathbf{U}_m + r\left[\mathbf{F}d\mathbf{n}_f(t) + \left(\mathbf{n}_f^{\mathsf{T}}(t)\otimes\mathbf{I}\right) d\mathrm{vec}\,\mathbf{F}\right] & (24)\end{aligned}$$

The entire system to be iterated consists of the equations (18) and (19) to project the populations of females and males, and the equations (22) and (24) to project the sensitivity of the female and male populations.

2.3 Dependent variables

From the derivatives of $\mathbf{n}(t)$ we can calculate the sensitivity of a diverse array of dependent variables, or summary statistics. An informal survey of Statistical Offices[1] finds that they typically present projections of the total population size, the proportional representation of specific age groups (e.g., working age, school-age children, retirement age, women of childbearing age), ratios such as the old-age, young-age, and total dependency ratios, and descriptors of the age distribution such as the median age in the population. The sensitivity of such statics is easily calculated, as follows.[2]

1. The sensitivity of total population size $N(t)$ is

$$\frac{dN(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathbf{1}^{\mathsf{T}}\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{25}$$

2. The sensitivity of a weighted population size: suppose that $N(t) = \mathbf{c}^{\mathsf{T}}\mathbf{n}(t)$, where $\mathbf{c}$ is a vector of weights (e.g., labor income of each age class, or the prevalence in each age class of some health condition). Then

$$\frac{dN(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathbf{c}^{\mathsf{T}}\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{26}$$

   where $\mathbf{c}$ is a vector of weights.

3. The sensitivity of a ratio of weighted population sizes (e.g. dependency ratios). Let

$$R(t) = \frac{\mathbf{a}^{\mathsf{T}}\mathbf{n}(t)}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} \tag{27}$$

[1] European Union, Germany, France, Belgium, Ireland, Estonia, Spain, Austria, Finland, Sweden, United Kingdom, Iceland, and Switzerland

[2] Many of these quantitites were discussed in a an ecological context in Caswell (2007)

where $\mathbf{a}$ and $\mathbf{c}$ are vectors of weights. be a ratio of weighted population sizes. Then (Caswell 2007)

$$\frac{dR(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \frac{dR(t)}{d\mathbf{n}^{\mathsf{T}}(t)} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{28}$$

$$= \left(\frac{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\mathbf{a}^{\mathsf{T}} - \mathbf{a}^{\mathsf{T}}\mathbf{n}(t)\mathbf{c}^{\mathsf{T}}}{(\mathbf{c}^{\mathsf{T}}\mathbf{n}(t))^2} \right) \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{29}$$

This formula captures many useful dependent variables, including the following.

(a) The proportional representation of an age group. In this case, $\mathbf{a}$ is a vector with ones to indicate the age group of interest and zeros elsewhere. The vector $\mathbf{c}$ is a vector of ones.

(b) Ratios such as the dependency ratio; in this case, $\mathbf{a}$ and $\mathbf{c}$ are both indicator vectors for the relevant age groups. The old-age dependency ratio, for example, is obtained by letting $\mathbf{a}$ indicate ages beyond retirement age and $\mathbf{c}$ indicate working ages.

(c) Weighted ratios. Instead of considering all individuals of retirement age, or working age, to be equal, $\mathbf{a}$ and $\mathbf{c}$ can be vectors of weights (e.g., labor income, consumption, deficit) for those age groups.

(d) Moments of the age distribution. The mean of the age distribution is obtained by setting

$$\mathbf{a} = \begin{pmatrix} 0.5 & 1.5 & 2.5 & \cdots \end{pmatrix}^{\mathsf{T}} \tag{30}$$

and setting $\mathbf{c}$ to be a vector of ones. The second moment of the age distribution is obtained by setting

$$\mathbf{a} = \begin{pmatrix} 0.5^2 & 1.5^2 & 2.5^2 & \cdots \end{pmatrix}^{\mathsf{T}} \tag{31}$$

and the variance in age is obtained from the first and second moments as usual.

(e) Mean and moments of age-specific properties. For example, suppose $B(x)$ is the mean body mass index (BMI) of age class x. Then the mean BMI in the population is obtained by setting $\mathbf{c} = \mathbf{1}$ and

$$\mathbf{a} = \begin{pmatrix} B(1) & B(2) & B(3) & \cdots \end{pmatrix}^{\mathsf{T}} \tag{32}$$

4. Short-term growth rates. Define the k-step growth rate, in weighted population size $\mathbf{c}^{\mathsf{T}}\mathbf{n}$, at time t as

$$\lambda(t) = \frac{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} \tag{33}$$

To obtain the sensitivity of $\lambda(t)$, note that

$$\frac{d\lambda(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \frac{\partial\lambda(t)}{\partial\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} \frac{d\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \frac{\partial\lambda(t)}{\partial\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)} \frac{d\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{34}$$

From (33), we have

$$\frac{\partial\lambda(t)}{\partial\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} = \frac{-\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)}{[\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)]^2} \tag{35}$$

$$\frac{\partial\lambda(t)}{\partial\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)} = \frac{1}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} \tag{36}$$

Assembling all the pieces gives the sensitivity of the short-term k-step growth rate,

$$\frac{d\lambda(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \frac{-\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)}{[\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)]^2}\mathbf{c}^{\mathsf{T}} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \frac{1}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)}\mathbf{c}^{\mathsf{T}} \frac{d\mathbf{n}(t+k)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{37}$$

In the special case where interest focuses on total population size, one simply sets $\mathbf{c} = \mathbf{1}$.

2.4 Perturbations by age and time

The sensitivity expressions (8), (22), and (24) provide information on perturbations applied to any age group at any time in the projection period. It may be appropriate to simplify this information, by aggregating sensitivity over age or time. For example,

1. The sensitivity of the population vector $\mathbf{n}(t)$ at time t to a perturbation, at time x, of all age classes by the same amount: this is given by

$$\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}\mathbf{1} \tag{38}$$

2. The sensitivity of the population vector at time t to a change in $\boldsymbol{\theta}$ applied equally at every time from $t = 0$ to $t = T$:

$$\sum_{x=0}^{T} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{39}$$

Many other dependent variables can be analyzed similarly (e.g., short term growth rates, temporal variances in population size, means and moments of the age distribution, etc.; see Caswell (2007)).

3 Projection of the population of Spain

To illustrate the use of matrix calculus techniques for sensitivity and elasticity calculations, we use a projection of the population of Spain, calculated and published by the Spanish Instituto Nacional de Estadstica (INE). The projection distinguishes single-year age groups (ages 0 to 100+ years) and sex of population members and covers a projection period from 2012 to 2052. Projection intervals have the length of one year (INE, 2012).

The projections use the cohort-component method and are based on assumptions about future fertility and mortality trends as well as future trends of international migration (i.e. migration crossing the external borders of Spain).

- Fertility assumptions are presented in the form of age-specific period rates. INE assumes that the total fertility rate will increase from 1.36 children per women in 2011 to 1.56 in 2051 and that the mean age of childbearing will rise from 31 to 32 years within the same projection period.

- Mortality assumptions are presented in the form of age-specific probabilities of dying (q(x)). The corresponding life expectancy values at birth for men are assumed to increase from 80 years in 2011 to 87 years in 2051, and from 83 years to 91 years for women.

- Migration assumptions are expressed in terms of age- and sex-specific immigration numbers and emigration probabilities. The latter are defined as the proportion of persons in each given age- and sex-specific group who are will emigrate in each projection year. INE assumes that the migratory balance of Spain, which was negative by 50.000 persons in 2011, will recover during the projection period. In the last ten projection years, the number of persons who move to Spain is assumed to exceed emigration numbers by around 438.000 persons. The emigration probabilities, however, are held constant over the entire projection interval.[3]

[3]This seems strange to us, but is clear in the data provided by INE.

On the basis of these assumptions, INE reports that the population of Spain will decrease in size from 46.2 million persons in 2012 to 41.5 million residents in 2052. We evaluated the sensitivity of these projection results by focusing on the female population of Spain only. In constructing the projection matrices $\mathbf{A}[t]$ we combined mortality and emigration as ways of leaving the population. Let P_i be the element in the $(i+1, i)$ entry of $\mathbf{A}$; then we write

$$P_i = (1 - q_i)(1 - r_i) \tag{40}$$

where q_i is the probability of death and r_i the probability of emigrating.

4 Some results

Here we present only a few of the results we have obtained from the analysis. Figure 1 shows the sensitivity of the entire age distribution at the terminal time $T = 40$, to changes in vital rates applied at every time. The highest sensitivities are to perturbations in mortality between ages 30 and 80. Changes in fetility at impossibly late ages would have large effects, but within the feasible range, the highest sensitivities are between ages 30–50. Perturbations in immigration have the greatest effects at ages 0–40.

Figure 2 collapses the age distribution to examine the sensitivity of total population size $N(t)$ for selected times ($t = 10, 20, 30, 40$) to changes in the vital rates applied at every time. The sensitivity to mortality rate is highest to perturbations at ages 40–60. The sensitivity to fertility is highest at ages 40–70. Changes in fertility at very late ages are impossible; within the range from 15–50 years, the sensitivity to fertility changes increases with age of the perturbation. Perturbations in immigration have the largest impact at earlier ages, from ages 20–30.

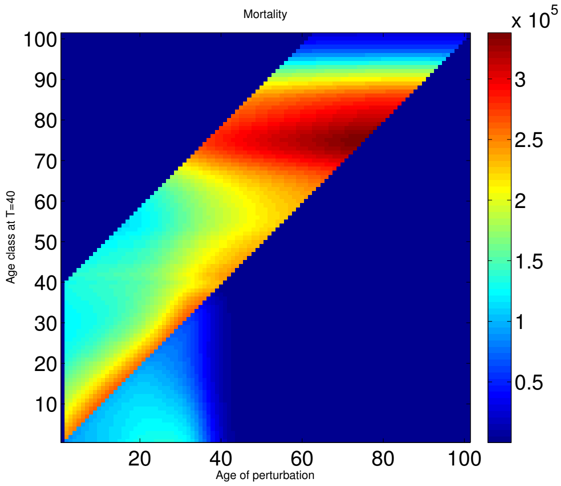
We calculate the sensitivity of the old-age and young-age dependency ratios to changes in mortality, fertility, and immigration, using (29), with young ages defined as age classes 1–15 and old ages defined as ages 65 and over. The results are shown in Figure 3, for several different observation times during the projection. Roughly speaking, increasing mortality at young ages would reduce the young-age dependency ratio and increase the old-age ratio, but the details are specific to this set of vital rates. Increasing fertility reduces the old-age dependency ratio and increases the young-age ratio.

Note that the sensitivities to changes in immigration are many orders of magnitude smaller than those to changes in mortality or fertility. This is because immigration is measured in numbers, while mortality and fertility are per capita rates. This would be a good place to use elasticity, rather than sensitivity, for comparisons among the rates.

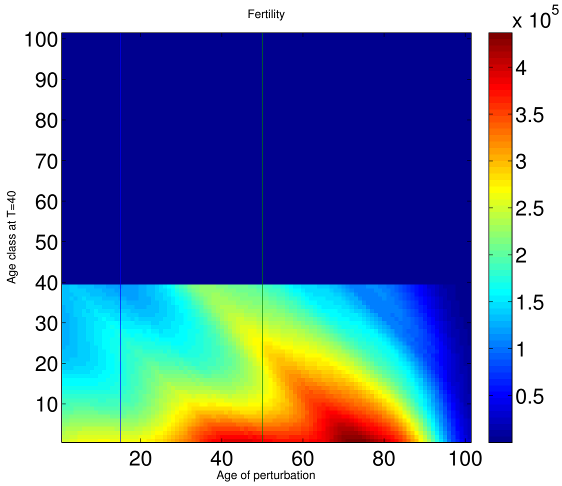
5 Literature cited

Institutio Nacional de Estadstica (2012): Proyeccin de la Poblacin de Espaa a Largo Plazo (2012-2052), Madrid .

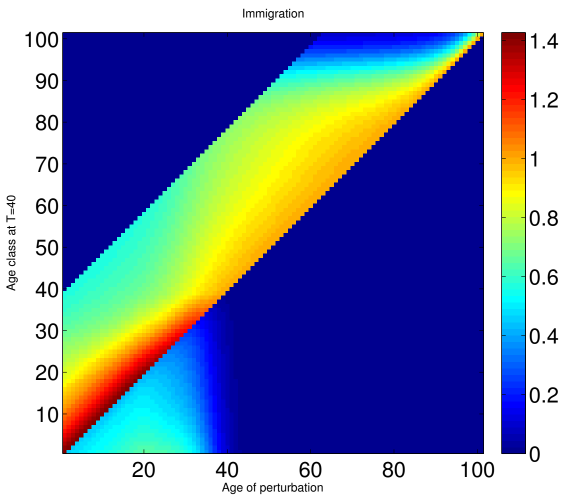
6 Figures



(a) Sensitivity to mortality

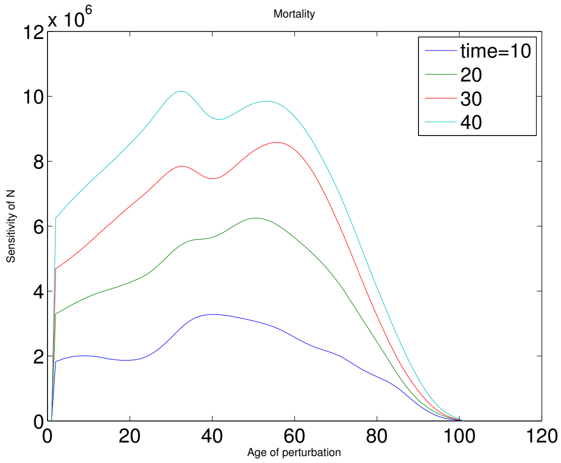


(b) Sensitivity to fertility

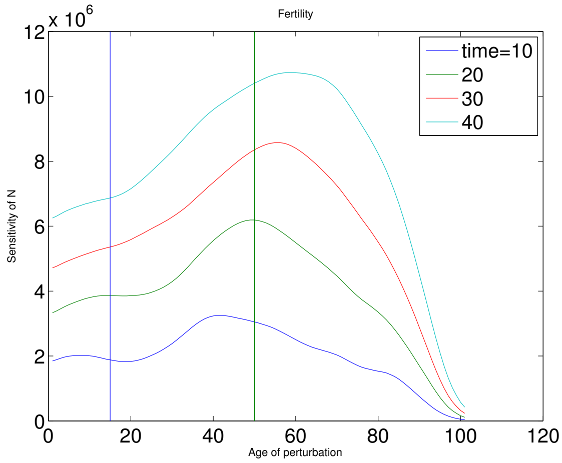


(c) Sensitivity to immigration

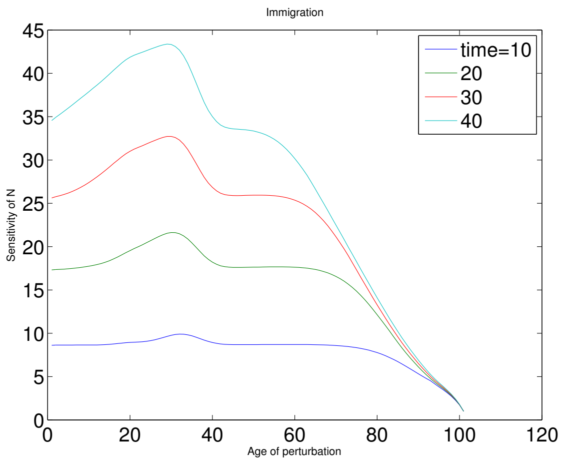
Figure 1: The sensitivity of $\mathbf{n}(T)$, where $T = 40$, to a change in age-specific vital rates, applied in every year from $t = 0$ to $t = T$. (a) mortality (sign reversed), (b) fertility, (c) immigration. Based on INE (2012) projections for Spain from 2012 to 2052.



(a) Sensitivity to mortality

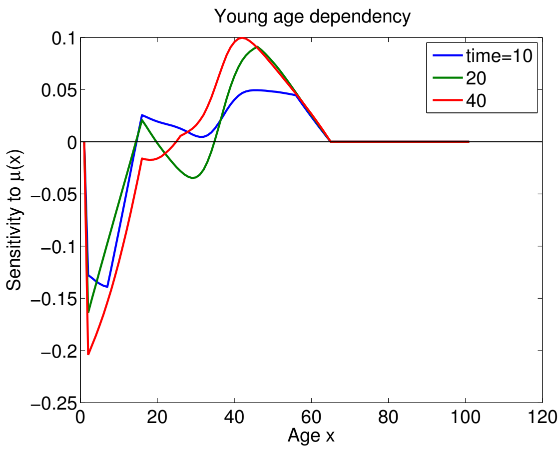


(b) Sensitivity to fertility

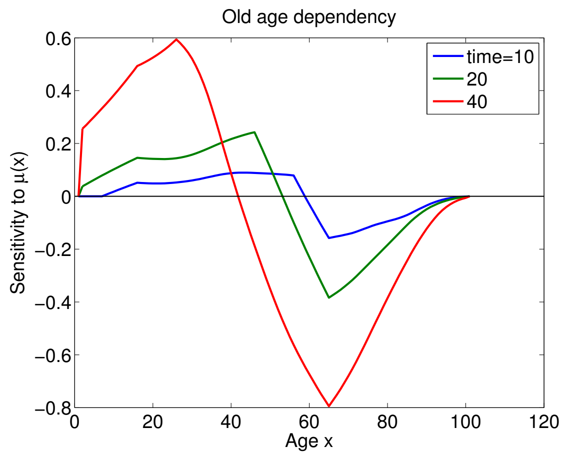


(c) Sensitivity to immigration

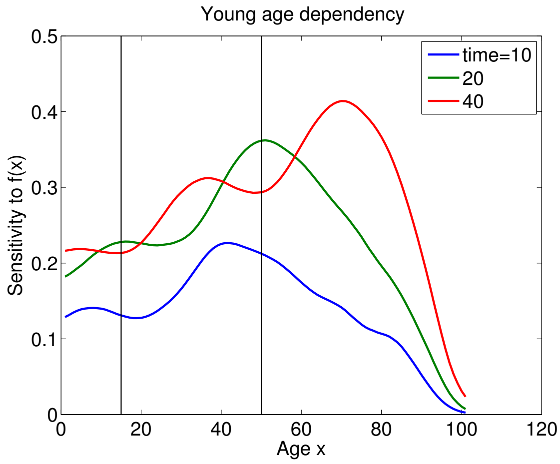
Figure 2: The sensitivity of total population size $N(t)$ to changes in age-specific vital rates, applied in every year from $t = 0$ to $t = T$. (a) mortality (sign reversed), (b) fertility (vertical lines indicate ages 15 and 50), (c) immigration. Based on INE (2012) projections for Spain from 2012 to 2052.



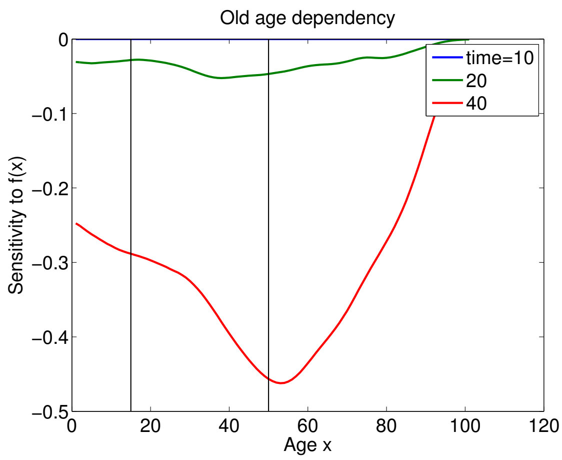
(a) Sensitivity to mortality



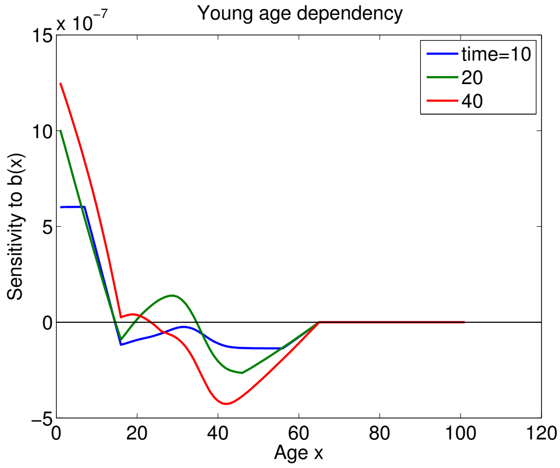
(b) Sensitivity to mortality



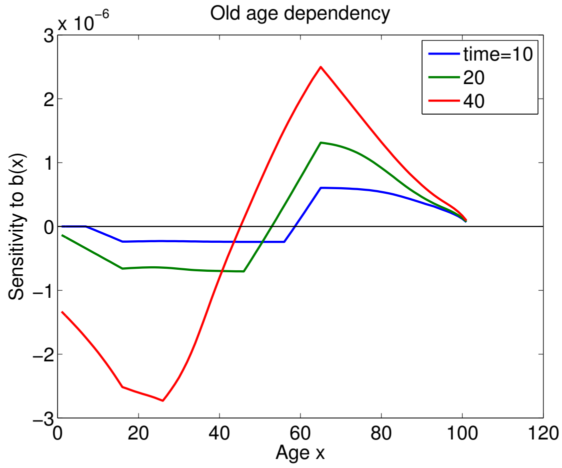
(c) Sensitivity to fertility



(d) Sensitivity to fertility



(e) Sensitivity to migration



(f) Sensitivity to migration

Figure 3: The sensitivity of the old-age and young-age dependency ratios to changes in age-specific mortality, fertility, or migration. The perturbations are applied in $t = 0$ to $t = T$. The vertical lines in panels (c) and (d) indicate ages 15 and 50. Based on INE (2012) projections for Spain from 2012 to 2052.