Renewal and stability in populations structured by remaining years of life

Tim Riffe
University of California, Berkeley

November 14, 2013

Abstract

We transform data classified by chronological age into data classified by remaining years of life (thanatological age). A model for population renewal and the corresponding projection matrix are presented for populations structured by thanatological age. Period results are derived using all available data from the HMD and HFD. We compare the intrinsic growth rate, r, as derived from the classic Lotka equation versus that derived on the basis of thanatological age. We also compare some transient indicators between the two models. Empirical results suggest that r from the thanatological model tends to be less erratic than Lotka's r, and the trajectory to stability tends to be faster with less oscillation.

*This is a work in progress. All findings are preliminary at this time, so please don't cite without permission of the author.

All demographic forces vary over age in known and regular ways. Information on such forces, along with the size and age structure of a population allows the demographer to make predictions about the future size and structure of a population. In this paper, we explore some formal demographic consequences of a particular redefinition of age. Instead of counting age as the time passed between birth and the present (or some other moment), consider age as the amount of time left from the present until death. Individuals in this case move in the same direction along imaginary life lines, but the reference point for age is now at the end of the life line instead of at the beginning. Of course, such information cannot be requested of survey respondents or register data, because one's time at death is typically unknown. Instead we approximate this perspective using a similar strategy to that used by Miller (2001) or Lee and Miller (2002), which uses lifetables and aggregated population counts.

Temporal perspectives and rescalings are abundant in demography already, but questions are left unanswered. Likely some demographic phenomena are best described as a function of time since birth, and others of time until death, while still others can be a function of both to some degree. This author supposes that fertility and reproduction can be a function of both age perspectives, and that the question of how to partition forces and events over time is a difficult one. Thanatological reproduction sits at the opposite extreme of chronological reproduction, the Lotka-Euler renewal model. By specifying a model of thanatological renewal, we both expand the toolbox available to approach complex temporal structure and we provide a set of broad findings to help understand the implications of temporal structure.

From chronological to thanatological age

Our point of departure is a little formula that gives the probability of dying at some age n years in the future, given survival to some earlier age, a. For age 0 this is simply the $d(a)$ column of the lifetable, standardized to sum to 1. For higher ages, we simply condition on survival:

$$f(n|a) = \mu(a+n)\frac{l(a+n)}{l(a)} \tag{1}$$

A convenient discretization of (1) is to simply work with the d_x column of the lifetable

$$d_{x+n,x} = \frac{d_{x+n}}{\sum_{i=x}^{\omega} d_i} \tag{2}$$

where ω is the highest age.[1] One produces in this way a probability density function over future death times for each age, and the $d_{x+n,x}$- weighted average of n for a given age x will simply produce the familiar remaining life expectancy column of the lifetable, e_x. Since $1 = \sum_{n=0}^{\omega-x} d_{x+n,x}$ for each x, we can use this to decompose age-classified demographic data. For example, we can break down a population pyramid into remaining years categories, like in Figure 1a, which shows this exercise for 2010 US data. The fun starts when we proceed to regroup the data by n instead of collapsing back to x. Now instead of remaining lifetime heterogeneity within each age group we have age heterogeneity within each discrete remaining years of life. Figure 1b shows this *re*composition for the same 2010 US data.

If P_x is an age-classified discrete population count, we can derive P_y, a remaining-years-classified population count, as follows:

$$P_y = \sum_{x=0}^{\omega} P_x \frac{d_{x+y} + d_{x+y+1}}{l_x + l_{x+1}} \tag{3}$$

and of course

$$P = \sum P_y = \sum P_a \tag{4}$$

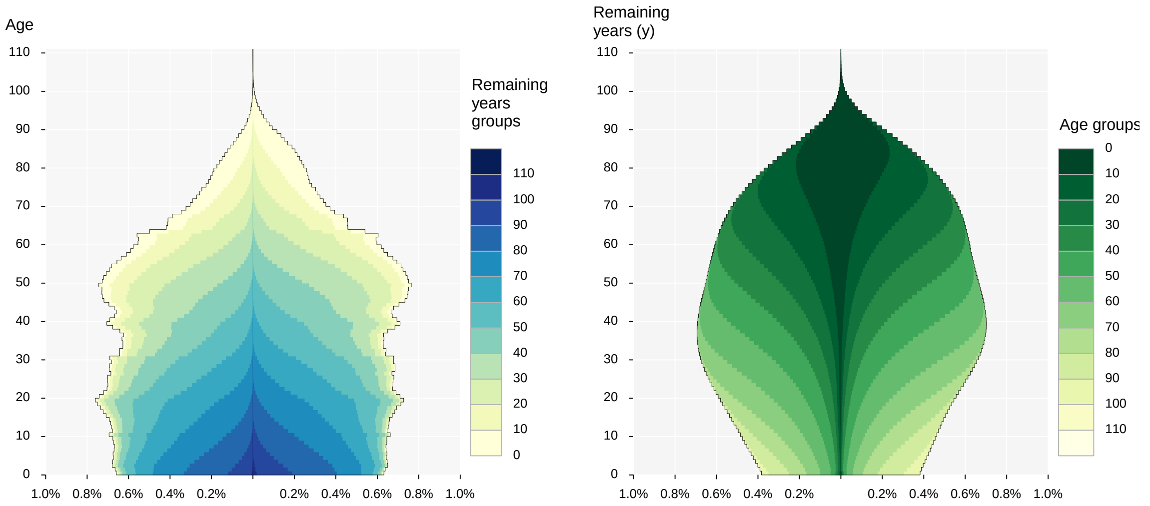
We refer to (3) as the thanatological age transformation[2], and this complements the notion of chronological age. It bears repeating that thanatological age is merely probabilistic, since it refers to the future according to particular assumptions about the survival curve. As such, Figure 1b is projective in nature, a forward-looking glance at a population given the age-structure and lifetable of a particular moment, whereas Figure 1a is reflective in nature, since a population's age-structure is mostly the fruit of past fertility, but also migration and to a lesser degree attrition. In a different paper we explore population structure by remaining years of life for its own sake. Note that any age-classified count can be reclassified in this way, as long as you have a lifetable that plausibly represents the population whose data you wish to reclassify. This means we can derive thanatological fertility rates, F_y, by applying (3) to age-classified birth counts, B_x, to get B_y and again to exposure-to-risk, E_x (here we take exposures from all ages), to get E_y and then dividing:

---

[1] In later formulas we just use l_x in the denominator to reduce clutter, but the above sum over d_x will eliminate rounding error in case one is working from a published life table.

[2] I have on good authority that Ken Wachter coined the term *thanatological* age.

Figure 1: 2010 US population



(a) Chronological age-structure decomposed by thanatological age.

(b) Thanatological age-structure decomposed by chronological age.

*Population and mortality data from the HMD.

$$F_y = \frac{B_y}{E_y} \tag{5}$$

We are interested to use thanatological fertility rates, F_y in models of population renewal, rather than to explicitly study the nature of these rates. Nonetheless it is best not to throw such rates into a model blindly, so we offer a schematic overview of some of their characteristics. Figure 2b represents the full variety in female thanatological period fertility rates that can be found for all years of data that overlap in the HFD and HMD.[3] Figure 2a gives ASFR for the same populations and years as a more familiar reference, but note that both scales are different! With no indication of particular populations or time series, one already concludes that thanatological fertility rates have a characteristic shape and are not random, erratic or informationless: there is a pattern to fertility rates by remaining years of life, and it varies over time and between populations within some range of normality. Thanatological fertility rates have a wider distribution than do chronological rates. Note that this spaghetti plot includes both fertility booms and busts, as well as some mortality crises (1918, WWII). Some patterns to note, but which we do not separate graphically here, are that the left tail (a gauge of how orphan-prone a population is) has tended to fall over time. All populations have shown a rightward shift in both the mean and modal female thanatological age at birth (TAB) over time (that's unambiguously a good thing). Several populations now show modal TABs of over 60 years. Thanatological total fertility rates track standard PTFR rather closely, but tend to be somewhat higher.

In general, such fertility rates will not be palatable for purposes of projection, unless the demographer believes that their empirical regularity is somehow stronger than typical ASFR — an argument we do not make. There may be alternative ways of defining thanatological fertility rates, for instance *after* stability, which would make this model component more conformable with the Lotka renewal model. First let us define the basic model.

The thanatological renewal model

For the population model that follows, we are most interested in using the kind of fertility rates shown in Figure 2b, and we refer to a unisex population. Assuming constant vital rates, the births for the present year are given by:

$$B(t) = \int_0^\infty P(y,t)F(y) \, \mathrm{d}y = \int_0^\infty P(a,t)F(a) \, \mathrm{d}a \tag{6}$$

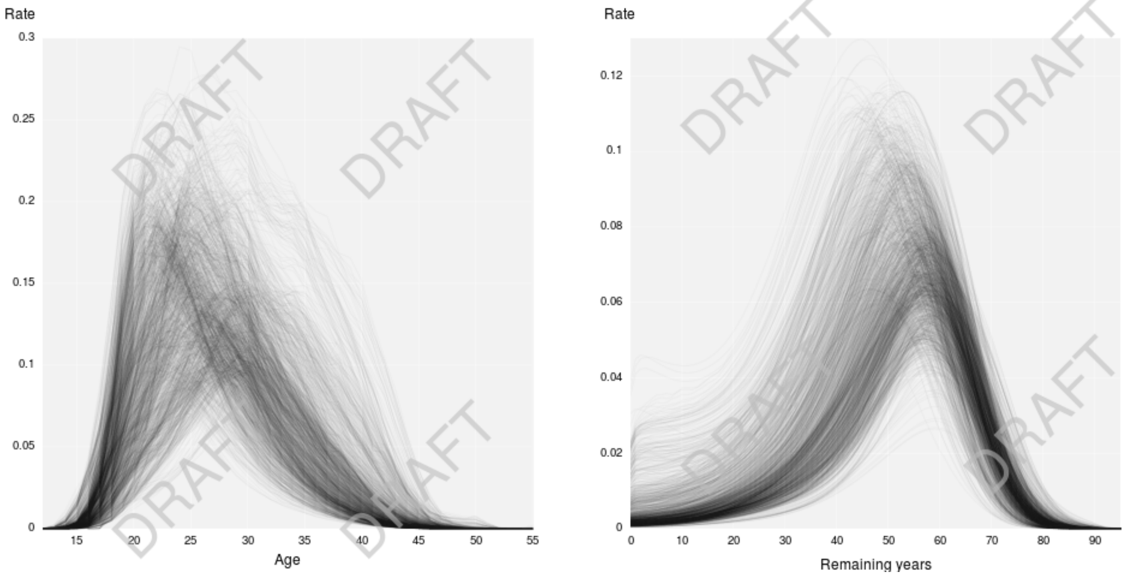
where $P(a), P(y)$ are population counts and $F(y), F(a)$ are exact specific fertility probabilities (rates), and $d(a)$ is the continuous lifetable death distribution with radix of 1, so $d(a)/l(0)$. The thanatological integral can be broken down back in terms of chronological age.

$$= \int_{y=0}^\infty \int_{a=0}^\infty F(y) \frac{P(a,t)d_(a+y)}{l(a)} \, \mathrm{d}a \, \mathrm{d}y \tag{7}$$

---

[3] There are as of this writing 1834 population-years of overlap between the HMD and HFD, including a wide variety of fertility and mortality combinations.

Figure 2: Chronological and thantological fertility rates, all 1600 country-year combinations present in both the HFD and HMD.* Note different x and y scales.



(a) Chronological period fertility rates.

(b) Thanatological period fertility rates.

DRAFT DRAFT DRAFT DRAFT DRAFT DRAFT DRAFT DRAFT

*AUT, 1951 – 2010; BGR, 1947 – 2009; BLR, 1964 – 2009; CAN, 1921 – 2007; CHE, 1932 – 2011; CZE, 1950 – 2011; DEUTE, 1956 – 2010; DEUTNP, 1990 – 2010; DEUTW, 1956 – 2010; ESP, 1922 – 2006; EST, 1959 – 2010; FIN, 1939 – 2009; FRATNP, 1946 – 2010; GBR_NIR, 1974 – 2009; GBR_NP, 1974 – 2009; GBR_SCO, 1945 – 2009; GBRTENW, 1938 – 2009; HUN, 1950 – 2009; IRL, 1955 – 2009; JPN, 1947 – 2009; LTU, 1959 – 2010; NLD, 1950 – 2009; NOR, 1967 – 2009; PRT, 1940 – 2009; RUS, 1959 – 2010; SVK, 1950 – 2009; SVN, 1983 – 2009; SWE, 1891 – 2010; TWN, 1976 – 2010; UKR, 1959 – 2006; USA, 1933 – 2010

We can relate the present population to past births with $P(a,t) = B(t-a)l(a)$.

$$= \int_{y=0}^{\infty} \int_{a=0}^{\infty} F(y)B(t-a)d(a+y) \, \mathrm{d}a \, \mathrm{d}y \tag{8}$$

And eventually– a bit quicker than is the case for chronological age– strong ergodicity will assert itself, and $B(t)$ will be related to $B(t-a)$ according to a constant factor e^{ra}, where r is the familiar intrinsic rate of growth.

$$= \int_{y=0}^{\infty} \int_{a=0}^{\infty} F(y)B(t)e^{-ra}d(a+y) \, \mathrm{d}a \, \mathrm{d}y \tag{9}$$

Divide out $B(t)$ to get back a familiar-looking renewal equation.

$$1 = \int_{y=0}^{\infty} \int_{a=0}^{\infty} F(y)d(a+y)e^{-ra} \, \mathrm{d}a \, \mathrm{d}y \tag{10}$$

Now compare this to Lotka's chronological formulation

$$1 = \int_{a=0}^{\infty} F(a)l(a)e^{-ra} \, \mathrm{d}a \tag{11}$$

and note that these are really quite similar, since $\int_{y=0}^{\infty} d(a+y) \, \mathrm{d}a = l(a)$. For intuition, notice that $l(a)$ is here split up into pieces of $d(a)$, and imagine a 2D surface of these, where one axis is chronological age and the other axis is thanatological age. For the chronological case (11), we multiply chronological age-specific fertility rates over one margin, and for the thanatological case over the other margin. There is a strong parallel here with the case of two-sex age-specific fertility rates. As in the case of divergence between male and female single-sex models, there will always be divergence between single-sex models under chronological versus thanatological age. Even though the modeled population stocks are in a way commensurable, the rates used, $F(a)$ versus $F(y)$ are calculated on the basis of differently distributed denominators $E(a)$ versus $E(y)$.

There are gaps in the above line of development, since the jump from (8) to (9) (strong ergodicity) is unproven, although it is rather intuitive, given the much greater density of connections within the model; persons from nearly any thanatologcal age can produce offspring that can have any other thanatological age. In this sense, the smoothing mechanism must be much stronger than that for chronoloical age, at least in most cases. A further conjecture is that the jump will also hold for the case of weak ergodicity. The mechanisms at play unfold in the same way as those so intuitively described by Arthur (1982), and said proof may apply here without further modification. A proof of the uniqueness of the solution to (10) is given in Appendix A.

Once one finds r from (10), other familiar stable population parameters can be calculated. For instance, we may calculate the mean thanatological generation time, T^y, as:

$$T^y = \frac{\int_{y=0}^{\infty} \int_{a=y}^{\infty} y e^{-ra} d(a) F(y) \, \mathrm{d}a \, \mathrm{d}y}{\int_{y=0}^{\infty} \int_{a=y}^{\infty} e^{-ra} d(a) F(y) \, \mathrm{d}a \, \mathrm{d}y} \tag{12}$$

The net reproduction rate, R_0 is related by, e.g.,

$$R_0 = e^{rT^y} \tag{13}$$

The birth rate, b, is given by

$$b = \frac{1}{\int_{y=0}^{\infty} \int_{a=y}^{\infty} e^{-ra} d(a) \, \mathrm{d}a \, \mathrm{d}y} \tag{14}$$

The stable age structure, c, where c_y is the proportion of the stable population with remaining years to live y, is given by

$$c(y) = b \int_{a=y}^{\infty} e^{-ra} d(a) \, \mathrm{d}a \tag{15}$$

Other stable population quantities may be estimated by similarly translating the various common definitions (e.g., in the glossary of Coale (1972)) to the present perspective. We will focus on the main model rather than on these.

The thanatological renewal model is coherent, but the idea may seem strange. What should one imagine under the model of thanatological population renewal? A useful mnemonic bases itself on Figures 1a and 1b. In the age-structured model, new generations appear at the bottom of the pyramid, and move up one rung per year. All ages are subject to decrement. In the thanatological *leaf*, each birth cohort increments to the population over the whole range of thanatological age according to $d(a)$, as seen in (10). Birth cohorts become the layers seen in Figure 1b. Each horizontal step is a death cohort, and these move one step down the pyramid each year without any decrement (indeed incrementing) until reaching the very bottom. In short, the locations of increment and decrement, and the direction of movement are all switched.

The thanatological projection matrix

These descriptions can be made more explicit by hashing out the projection matrix that corresponds to the thanatological renewal model. As with the age-structured Leslie matrix, the thanatological projection matrix, $\mathbf{Y}$, is square and of dimension $n \times n$, where n is the number of remaining-years classifications into which the population is divided. The matrix contains elements for survival and elements for fertility. Unlike Leslie matrices, $\mathbf{Y}$ is not sparse, but is populated primarily with non-zero entries.

Of interest is that mortality occurs in only the population class with zero remaining years of life. Thanatological age 1 in year t moves to 0 in year $t+1$. Thus, instead of in the subdiagonal, we place survival in the superdiagonal. All survival values are 1, since there is no decrement, and the upper-left corner of this superdiagonal contains a 0, for full decrement. We illustrate using a 6×6 matrix. The survival component of $\mathbf{Y}$ is organized as in Matrix 0.1.

Matrix 0.1: Survival component of unisex thanatological projection matrix, $\mathbf{Y}$

$$\begin{array}{c c}
e_y & \begin{array}{cccccc} 0_t & 1_t & 2_t & 3_t & 4_t & 5_t \end{array} \\
\begin{array}{c} 0_{t+1} \\ 1_{t+1} \\ 2_{t+1} \\ 3_{t+1} \\ 4_{t+1} \\ 5_{t+1} \end{array} &
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

Fertility inputs to the matrix are derived from single-sex thanatological fertility and the lifetable d_a distribution, where a indexes age, but is equal to y, remaining years of life for persons aged 0. Fertility in a thanatologically structured population occurs in all but the highest remaining years classes. For our example, say that fertility is observed in classes 0-4, while the final class has no fertility, where F_y indicates the fertility probability for class y in the year t entering population (in the matrix columns). Each F_y is then distributed according to d_a with no further translation, since the d_a lifetable column already refers to age 0. Thus, the fertility entry in row m and column n of $\mathbf{Y}$ will be $F_n \cdot d_m$. We assume that those dying over the course of year t (the first column) are exposed to fertility for half of the year,[4] and so discount the fertility entry accordingly. Further, infant mortality, $F_y \cdot d_0$, located in the first row, must also be discounted, since part of the mortality will occur in the same year t and the rest in year $t+1$. The first row of fertility must be further discounted by a factor, λ, in order to account for the fact that infant mortality is higher in the lower Lexis triangle than in the upper: of those infants who die in the first year of life, a proportion equal to

---

[4] One might be tempted to not allow for fertility at all for females dying in year t, but recall that fertility is measured in the moment of birth, and not conception.

λ do not make it to December 31st of the calendar year in which they were born.[5] The fertility component of $\mathbf{Y}$ is then composed as in Matrix 0.2.

Matrix 0.2: Fertility component of unisex thanatological projection matrix, $\mathbf{Y}$

$$\begin{array}{c} e_y \\ 0_{t+1} \\ 1_{t+1} \\ 2_{t+1} \\ 3_{t+1} \\ 4_{t+1} \\ 5_{t+1} \end{array} \begin{array}{c} \begin{array}{cccccc} 0_t & 1_t & 2_t & 3_t & 4_t & 5_t \end{array} \\ \begin{pmatrix} (1-\lambda)\frac{F_0 d_0}{2} & (1-\lambda)F_1 d_0 & (1-\lambda)F_2 d_0 & (1-\lambda)F_3 d_0 & (1-\lambda)F_4 d_0 & 0 \\ \frac{F_0 d_1}{2} & F_1 d_1 & F_2 d_1 & F_3 d_1 & F_4 d_1 & 0 \\ \frac{F_0 d_2}{2} & F_1 d_2 & F_2 d_2 & F_3 d_2 & F_4 d_2 & 0 \\ \frac{F_0 d_3}{2} & F_1 d_3 & F_2 d_3 & F_3 d_3 & F_4 d_3 & 0 \\ \frac{F_0 d_4}{2} & F_1 d_4 & F_2 d_4 & F_3 d_4 & F_4 d_4 & 0 \\ \frac{F_0 d_5}{2} & F_1 d_5 & F_2 d_5 & F_3 d_5 & F_4 d_5 & 0 \end{pmatrix} \end{array}$$

The survival and fertility components of $\mathbf{Y}$ add together elementwise, thus the full 6×6 matrix is composed as in Matrix 0.3.

Matrix 0.3: A full unisex thanatological projection matrix, $\mathbf{Y}$

$$\mathbf{Y} = \begin{array}{c|cccccc} e_y & 0_t & 1_t & 2_t & 3_t & 4_t & 5_t \\ \hline 0_{t+1} & (1-\lambda)\frac{F_0 d_0}{2} & (1-\lambda)F_1 d_0 + 1 & (1-\lambda)F_2 d_0 & (1-\lambda)F_3 d_0 & (1-\lambda)F_4 d_0 & 0 \\ 1_{t+1} & \frac{F_0 d_1}{2} & F_1 d_1 & F_2 d_1 + 1 & F_3 d_1 & F_4 d_1 & 0 \\ 2_{t+1} & \frac{F_0 d_2}{2} & F_1 d_2 & F_2 d_2 & F_3 d_2 + 1 & F_4 d_2 & 0 \\ 3_{t+1} & \frac{F_0 d_3}{2} & F_1 d_3 & F_2 d_3 & F_3 d_3 & F_4 d_3 + 1 & 0 \\ 4_{t+1} & \frac{F_0 d_4}{2} & F_1 d_4 & F_2 d_4 & F_3 d_4 & F_4 d_4 & 1 \\ 5_{t+1} & \frac{F_0 d_5}{2} & F_1 d_5 & F_2 d_5 & F_3 d_5 & F_4 d_5 & 0 \end{array}$$

Thanatological age-classes will ideally terminate at the highest value permitted by data. For the data used here, there are 111 total age classes, which translate to 111 total remaining-years classes (0-110+). In practice $\mathbf{Y}$ becomes a 111×111 matrix, with most entries non-zero. Construction may appear tedious for this reason. However, note that the bulk of fertility entries can be derived as the outer (tensor) product $d_a \otimes f_y$, leaving only the first row and first column mortality discounting followed by the addition of the survival superdiagonal. In most statistical programming languages constructing $\mathbf{Y}$ entails only a couple more lines of code than constructing a Leslie matrix.

---

[5] λ can be derived directly from death counts data classified by Lexis triangles. In the US, λ has behaved similarly for males and females, falling steadily from around 0.9 in 1969 to 0.86 around 1990, since which time it has steadily risen to around 0.87. That is to say, λ has varied, but not drastically. These numbers are just meant to give a feel for the ranges that λ can be expected to receive. If the demographer does not have information to derive λ directly, ad hoc or semidirect methods may be used to assign a reasonable proportion.

As with Leslie matrices, the above projection matrix may be manipulated using standard matrix techniques. Where $\mathbf{p}$ is our population vector, we project by multiplying $\mathbf{Y}$ from the left: $\mathbf{p}(t+1) = \mathbf{Y}\mathbf{p}(t)$. From $\mathbf{Y}$, we can extract such information as the intrinsic growth rate, r (natural log of the largest real eigenvalue), the stable thanatological age-structure (the real part of the eigenvector that corresponds to the largest real eigenvalue), or the pace of convergence to stability (the ratio of the 1st to the 2nd eigenvalues). [6]

Some empirical findings

The renewal equation and the discrete projection matrix can be put to work with data. At this time we are only ready to report some early results, and we do not yet have explanations some of the patterns that we report. Of our 1834 population-years, we optimize r from both (10) and (11)[7]. In this sample, both versions of r are only plausibly equal in a single instance. Usually thantological r is greater than Lotka's r (1373 cases). When Lotka's r is positive (693 cases), thanatological r is greater just over of 50% of the time (356), but when Lotka's r is negative, thanatological r is the greater of the two around 90% of the time (1017 cases). These two approximations of r are of opposite sign 138 times. We provide a comparison of the r distributions in Figure 3. Mean locations for each distribution are indicated with vertical dashed lines; thano. -0.0010; chrono. -0.0027. The distribution of thanatological r is more compact than, with the ratio of variances (thano./chrono.) of about 0.75. Usually the two theoretical values of r move in the same direction, but thanatological r is over time the less erratic of the two, and it usually paints a less dire picture when both are negative.

From the projection matrix, the ratio of the largest to the second largest eigenvalue, the damping ratio, is an indicator of the springiness of the stable population structure, c_y or c_a, with respect to disturbances from the stable state as determined by vital rate distributions. A higher damping ratio means that the population structure oscillates back to its stable state faster, i.e., that oscillations decrease in size more rapidly. The thanatological damping ratio was greater than the Leslie damping ratio for all 1834 population-years included in our empirical analysis. Leslie damping ratios ranged from 1.01258 to 1.0518, while thanatological damping ratios ranged from 1.0455 to 1.0868.

* this is a work in progress, and by the time the EPC comes around I'll have shifted the focus away on comparing stable results estimated from data (the last couple figures), and more toward the reversibility of chronological and thanatological stable age structures, as well as a comparative eigen-analysis of the Leslie and thanatological projection matrices.

A Unique solution for thanatological r

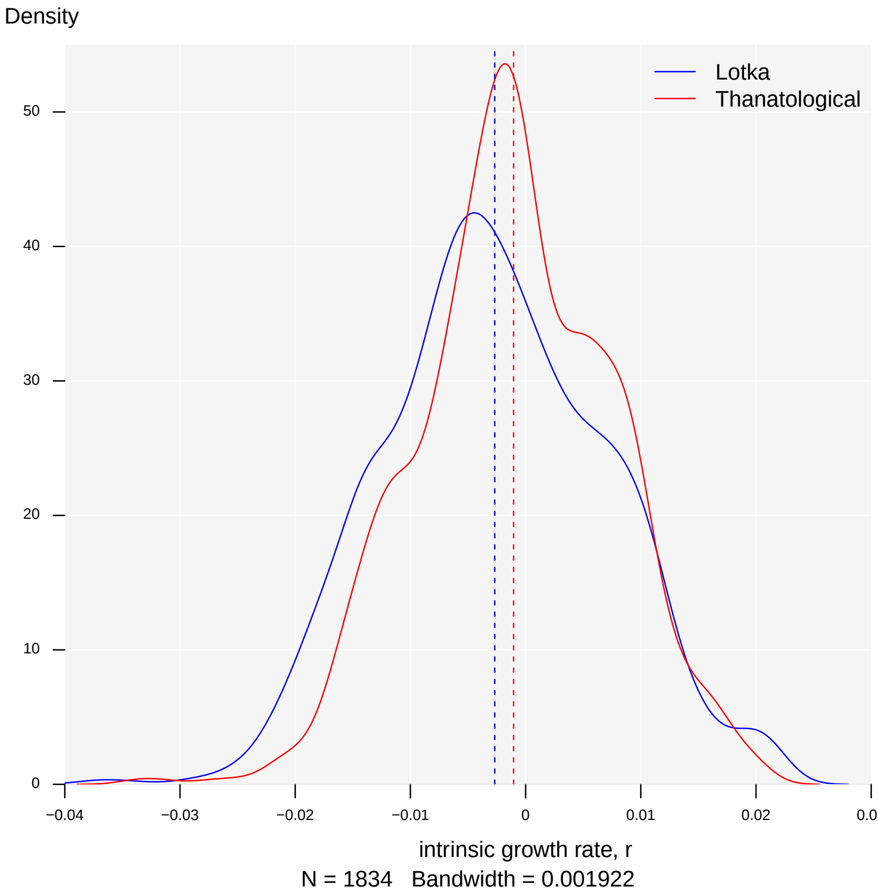
The solution for the intrinsic growth rate, r, is unique for the case of the thanatological renewal model, (10), and can be proven so in essentially the same fashion as those in existence for the Lotka-Euler model, (11).

---

[6] See Caswell (2001, p.86-87).

[7] It is possible to optimize r using a variety of approximations, or by using a generic optimizer. We have used the method proposed by Coale (1957) for Lotka's r and a modified version of the same for the thanatological r. Details and / or implementation code available on request.

Figure 3: Distribution of r, chronological (Lotka) and thanatological*.



* Data from HMD and HFD. Countries and years listed in Figure 2.

This little proof follows that given in *pressat1972demographic*. Define a convenience function, $I(r)$, for the integrand of (10) for a given r and fixed $F(y)$ and $d(a)$:

$$I(r) = \int_{y=0}^{\infty} \int_{a=0}^{\infty} F(y) d(a+y) e^{-ra} \, \mathrm{d}a \, \mathrm{d}y \tag{16}$$

Since the death distribution function, $d(a)$, and fertility function, $F(y)$, are continuous and non-negative, $\lim_{r\to+\infty} I(r) = 0$ and $\lim_{r\to-\infty} I(r) = \infty$. If $r_2 > r_1$, then $I(r_1) > I(r_2)$. $I()$ is therefore a continuous and monotonically decreasing function of r with boundaries that include the value 1 of (10), and necessarily only obtain this value once.

References

W Brian Arthur. The ergodic theorems of demography: a simple proof. *Demography*, 19(4):439–445, 1982.

H. Caswell. *Matrix population models: construction, analysis, and interpretation.* Sinaur Associates, Inc. Publishers, 2001.

A.J. Coale. A New Method for Calculating Lotka's r–the Intrinsic Rate of Growth in a Stable Population. *Population studies*, 11(1):92–94, 1957. ISSN 0032-4728.

Ansley J Coale. *The growth and structure of human populations: A mathematical investigation.* Princeton University Press Princeton, NJ, 1972.

Max Planck Institute for Demographic Research (Germany) and Vienna Institute of Demography (Austria). Human fertility database. online, 09 2014. Available at www.humanfertility.org (data downloaded on [Sept 1, 2013]).

Ronald Lee and Timothy Miller. An approach to forecasting health expenditures, with application to the us medicare system. *Health Services Research*, 37(5):1365–1386, 2002.

Tim Miller. Increasing longevity and medicare expenditures. *Demography*, 38(2):215–226, 2001.

University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Human Mortality Database. online, Sep 2013. URL `www.mortality.org`. (data downloaded in September, 2013).